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Qualitative aspects of entanglement in the Jaynes–Cummings model with an external quantum field

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Abstract

We present a mathematical procedure which leads us to obtain analytical solutions for the atomic inversion and Wigner function in the framework of the Jaynes–Cummings model with an external quantum field, for any kind of cavity and driving fields. Such solutions are expressed in the integral form, with their integrands having a common term that describes the product of the Glauber–Sudarshan quasiprobability distribution functions for each field, and a kernel responsible for the entanglement. Considering two specific initial states of the tripartite system, the formalism is then applied to calculate the atomic inversion and Wigner function where, in particular, we show how the detuning and amplitude of the driving field modify the entanglement. In addition, we also obtain the correct quantum-mechanical marginal distributions in phase space.

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1. Introduction

The concept of entanglement naturally appears in quantum mechanics when the superposition principle is applied to composite systems. In this sense, a multipartite system is entangled when their physical properties cannot be described through a tensor product of density operators associated with their different parts which constitute the whole system. An immediate consequence of this important effect has its origin in the theory of quantum measurement [1]: the entangled state of the multipartite system can reveal information about its constituent

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parts. However, this information is extremely sensitive to the dissipative coupling between the macroscopic meter and its environment. In fact, entangled states involving macroscopic meters are rapidly transformed into statistical mixtures of product states and this fast relaxation process characterizes the decoherence [2–4]. According to Raimond *et al* [5]: 'the decoherence itself involves entanglement since the meter gets entangled with its environment. As the information leaks into the environment, the meter's state is obtained by tracing over the environment variables, leading to the final statistical mixture. This analysis is fully consistent with the Copenhagen description of a measurement.' Beyond these fundamental features, entangled states have potential applications for quantum information processing and quantum computing [6–9], quantum teleportation [10], dense coding [11] and quantum cryptographic schemes [12].

A feasible physical system to generate entangled states is given by the Jaynes–Cummings model (JCM) which describes the matter–field interaction [13, 14]. It is typically realized in cavity QED experiments involving Rydberg atoms crossing superconducting cavities (one by one) in different frequency regimes and configurations, with relaxation rates small and well understood [5]. Recently, many authors have investigated the two-mode and driven JCM in different contexts and predicted new interesting results [15–27]. For instance, Abdalla *et al* [24] have introduced a new model of Hamiltonian which describes the interaction between a two-level atom and two electromagnetic fields injected simultaneously within a cavity, and obtained an exact expression for the wavefunction in the Schrödinger picture by means of a canonical transformation. Consequently, this exact wavefunction was used in order to investigate the atomic inversion and the degree of entanglement of this new model. Solano *et al* [26] have proposed a method of generating multipartite entanglement through the interaction of a system of N two-level atoms in a cavity of high quality factor with a strong classical driving field. Following the authors, the main advantage of this external field in the system under consideration is the great flexibility in generating entangled states, since it provides freedom to choose the detuning and strength of the field. On the other hand, Wildfeuer and Schiller [27] have used Schwinger's oscillator model to obtain a mathematical solution for the generation of entangled N-photon states in the framework of the two-mode JCM. Here we develop a mathematical procedure which permits us to obtain compact solutions for atomic inversion and the Wigner function in the framework of the JCM with an external quantum field, considering any cavity and external fields. In particular, both solutions are expressed in the integral form with their integrands presenting a common term that describes the product of the Glauber–Sudarshan quasiprobability distributions [28] for each field, and a kernel responsible for the correlations. To illustrate our results we fix the cavity field in the even- and odd-coherent states [29], and the external field in the coherent state. Furthermore, we show how the detuning and amplitude of the external field modify the entanglement in the tripartite system via the Wigner function.

The paper is organized as follows. In section 2 we obtain the time-evolution operator and the matrix elements of the density operator for the JCM with an external quantum field, being the cavity and external fields described in the diagonal representation of coherent states. Following, we fix the cavity field in the even- and odd-coherent states and the external field in the coherent state to investigate, in section 3, the effects of amplitude of the external field and detuning parameters upon the atomic inversion. In section 4 we derive a formal expression for the Wigner function associated with the cavity field which permits us to analyse how the entanglement is modified in the tripartite system. Moreover, we also obtain analytical expressions for the correct quantum-mechanical marginal distributions in phase space. Section 5 contains our summary and conclusions. Finally, appendices A and B contain the main steps to calculate the atomic inversion and Wigner function, respectively.

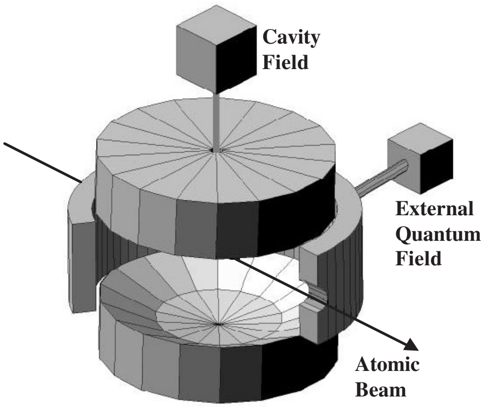

Figure 1. Experimental apparatus used in the description of the JCM with an external quantum field.

2. Algebraic aspects of the JCM with an external quantum field

In general, the JCM with an external field consists of a two-level atom interacting nonresonantly with a single-mode cavity field, and driven additionally by an external quantum field through one open side of the cavity (the experimental scheme in the context of cavity QED is sketched in figure 1). Within the dipole and rotating-wave approximations, the dynamics of the atom–cavity system is governed by the Hamiltonian $\mathbf{H} = \mathbf{H}_0 + \mathbf{V}$, where

$$\mathbf{H}_0 = \hbar\omega(\mathbf{a}^\dagger\mathbf{a} + \mathbf{b}^\dagger\mathbf{b}) + \tfrac{1}{2}\hbar\omega\boldsymbol{\sigma}_{\mathrm{z}} \tag{1}$$

$$\mathbf{V} = \tfrac{1}{2}\hbar\delta\boldsymbol{\sigma}_{\mathrm{z}} + \hbar\kappa_{\mathrm{a}}(\mathbf{a}^\dagger\boldsymbol{\sigma}_- + \mathbf{a}\boldsymbol{\sigma}_+) + \hbar\kappa_{\mathrm{b}}(\mathbf{b}^\dagger\boldsymbol{\sigma}_- + \mathbf{b}\boldsymbol{\sigma}_+). \tag{2}$$

Here, ω is the cavity field frequency (we assume the resonance condition between the cavity and external fields), ω_0 is the atomic transition frequency, $\delta = \omega_0 - \omega$ is the detuning frequency, and $\kappa_{\mathrm{a(b)}}$ is the coupling constant between the atom and the cavity (external) field. The atomic spin-flip operators $\boldsymbol{\sigma}_\pm$ and $\boldsymbol{\sigma}_{\mathrm{z}}$ are defined as $\boldsymbol{\sigma}_+ = |e\rangle\langle g|$, $\boldsymbol{\sigma}_- = |g\rangle\langle e|$ and $\boldsymbol{\sigma}_{\mathrm{z}} = |e\rangle\langle e| - |g\rangle\langle g|$ ($|g\rangle$ and $|e\rangle$ correspond to the ground and excited states of the atom), with the following commutation relations: $[\boldsymbol{\sigma}_{\mathrm{z}}, \boldsymbol{\sigma}_\pm] = \pm 2\boldsymbol{\sigma}_\pm$ and $[\boldsymbol{\sigma}_+, \boldsymbol{\sigma}_-] = \boldsymbol{\sigma}_{\mathrm{z}}$. Furthermore, $\mathbf{a}\left(\mathbf{a}^\dagger\right)$ and $\mathbf{b}\left(\mathbf{b}^\dagger\right)$ are the annihilation (creation) operators of the single-mode cavity and external fields, respectively. It is important to mention that the quantum nature of the fields used in many proposed schemes for quantum information processing present serious consequences in large scale quantum computations, since the uncertainty principle and the possibility of becoming entangled with the physical qubits represent possible limitations on quantum computing [30–34]. In this sense, van Enk and Kimble [31] have considered the interaction of atomic qubits laser fields and quantify atom–field entanglement in various situations of interest where, in particular, they found that the entanglement decreases with the mean number of photons $\langle\mathbf{n}\rangle$ in a laser beam as $E \propto \langle\mathbf{n}\rangle^{-1}\log_2\langle\mathbf{n}\rangle$ for $\langle\mathbf{n}\rangle \gg 1$. Pursuing this line, Gea-Banacloche [32] has investigated the quantum nature of the laser fields used in the manipulation of quantum information, focusing especially on phase errors and their effects on error-correction schemes (for more details, see [33, 34]).

Now, let us define the quasi-mode operators $\mathbf{A} = \epsilon_{\mathrm{a}}\mathbf{a} + \epsilon_{\mathrm{b}}\mathbf{b}$ and $\mathbf{B} = \epsilon_{\mathrm{b}}\mathbf{a} - \epsilon_{\mathrm{a}}\mathbf{b}$ (where $\epsilon_{\mathrm{a(b)}} = \kappa_{\mathrm{a(b)}}/\kappa_{\mathrm{eff}}$, and $\kappa_{\mathrm{eff}}^2 = \kappa_{\mathrm{a}}^2 + \kappa_{\mathrm{b}}^2$ is an effective coupling constant), which satisfy the commutation relations

$$\begin{array}{lll}[\mathbf{A},\mathbf{A}^\dagger]=\mathbf{1} & [\mathbf{N}_\mathrm{A},\mathbf{A}]=-\mathbf{A} & [\mathbf{N}_\mathrm{A},\mathbf{A}^\dagger]=\mathbf{A}^\dagger\\ [\mathbf{B},\mathbf{B}^\dagger]=\mathbf{1} & [\mathbf{N}_\mathrm{B},\mathbf{B}]=-\mathbf{B} & [\mathbf{N}_\mathrm{B},\mathbf{B}^\dagger]=\mathbf{B}^\dagger\\ [\mathbf{A},\mathbf{B}]=0 & [\mathbf{A},\mathbf{B}^\dagger]=0 & [\mathbf{N}_\mathrm{A},\mathbf{N}_\mathrm{B}]=0\end{array}$$

$\mathbf{N}_\mathrm{A}=\mathbf{A}^\dagger\mathbf{A}(\mathbf{N}_\mathrm{B}=\mathbf{B}^\dagger\mathbf{B})$ being the number operator related to the quasi-mode operator $\mathbf{A}(\mathbf{B})$. Introducing the number-sum operator $\mathbf{S}=\mathbf{N}_\mathrm{A}+\mathbf{N}_\mathrm{B}$ and the number-difference $\mathbf{D}=\mathbf{N}_\mathrm{A}-\mathbf{N}_\mathrm{B}$, we verify that:

(i) $\mathbf{S}=\mathbf{n}_\mathrm{a}+\mathbf{n}_\mathrm{b}$ is a conserved quantity ($\mathbf{n}_\mathrm{a}=\mathbf{a}^\dagger\mathbf{a}$ and $\mathbf{n}_\mathrm{b}=\mathbf{b}^\dagger\mathbf{b}$ are the photon-number operators of the cavity and external fields);

(ii) the operator $\mathbf{D}$ can be written in terms of the generators $\{\mathbf{K}_+,\mathbf{K}_-,\mathbf{K}_0\}$ of the $su(2)$ Lie algebra,

$$\mathbf{D}=2\left(\epsilon_\mathrm{a}^2-\epsilon_\mathrm{b}^2\right)\mathbf{K}_0+2\epsilon_\mathrm{a}\epsilon_\mathrm{b}\left(\mathbf{K}_-+\mathbf{K}_+\right) \tag{3}$$

where $\mathbf{K}_-=\mathbf{a}\mathbf{b}^\dagger$, $\mathbf{K}_+=\mathbf{a}^\dagger\mathbf{b}$ and $\mathbf{K}_0=\frac{1}{2}\left(\mathbf{a}^\dagger\mathbf{a}-\mathbf{b}^\dagger\mathbf{b}\right)$, with $[\mathbf{K}_-,\mathbf{K}_+]=-2\mathbf{K}_0$ and $[\mathbf{K}_0,\mathbf{K}_\pm]=\pm\mathbf{K}_\pm$;

(iii) the commutation relation between the operators $\mathbf{S}$ and $\mathbf{D}$ is null, i.e., $[\mathbf{S},\mathbf{D}]=0$; and consequently

(iv) the Hamiltonian $\mathbf{H}$ simplifies to

$$\mathbf{H}_0=\hbar\omega\mathbf{S}+\tfrac{1}{2}\hbar\omega\boldsymbol{\sigma}_\mathrm{z} \tag{4}$$

$$\mathbf{V}=\tfrac{1}{2}\hbar\delta\boldsymbol{\sigma}_\mathrm{z}+\hbar\kappa_\mathrm{eff}(\mathbf{A}^\dagger\boldsymbol{\sigma}_-+\mathbf{A}\boldsymbol{\sigma}_+) \tag{5}$$

with $[\mathbf{H}_0,\mathbf{V}]=0$. This fact leads us to obtain the Hamiltonian $\mathbf{H}_\mathrm{int}=\mathbf{V}$ in the interaction picture, which describes the well-known nonresonant JCM Hamiltonian for an atom interacting with the quasi-mode $\mathbf{A}$, and whose coupling constant is given by κ_eff. Thus, the unitary time-evolution operator is the usual nonresonant JCM time-evolution operator.

If one considers the time-evolution operator $\mathcal{U}(t)=\exp(-\mathrm{i}\mathbf{V}t/\hbar)$ of the atom–cavity system written in the atomic basis, the elements $\mathcal{U}_{ij}(t)$ of the 2×2 matrix can be expressed as [35]

$$\mathcal{U}_{11}(t)=\cos(t\sqrt{\beta_\mathrm{A}})-\mathrm{i}\frac{\delta}{2}\frac{\sin(t\sqrt{\beta_\mathrm{A}})}{\sqrt{\beta_\mathrm{A}}} \tag{6}$$

$$\mathcal{U}_{12}(t)=-\mathrm{i}\,\kappa_\mathrm{eff}\frac{\sin(t\sqrt{\beta_\mathrm{A}})}{\sqrt{\beta_\mathrm{A}}}\mathbf{A} \tag{7}$$

$$\mathcal{U}_{21}(t)=-\mathrm{i}\,\kappa_\mathrm{eff}\mathbf{A}^\dagger\frac{\sin(t\sqrt{\beta_\mathrm{A}})}{\sqrt{\beta_\mathrm{A}}} \tag{8}$$

$$\mathcal{U}_{22}(t)=\cos(t\sqrt{\varphi_\mathrm{A}})+\mathrm{i}\frac{\delta}{2}\frac{\sin(t\sqrt{\varphi_\mathrm{A}})}{\sqrt{\varphi_\mathrm{A}}} \tag{9}$$

where $\varphi_\mathrm{A}=\kappa_\mathrm{eff}^2\mathbf{N}_\mathrm{A}+(\delta/2)^2\mathbf{1}$ and $\beta_\mathrm{A}=\varphi_\mathrm{A}+\kappa_\mathrm{eff}^2\mathbf{1}$. This result permits us to determine the density operator $\rho(t)=\mathcal{U}(t)\rho(0)\mathcal{U}^\dagger(t)$, $\rho(0)$ being the density operator of the system at time $t=0$. For convenience in the calculations, we assume the atom is initially in the excited state and the cavity and external fields are in the diagonal representation of coherent states, i.e., $\rho(0)=\rho_\mathrm{at}(0)\otimes\rho_\mathrm{ab}(0)$ with $\rho_\mathrm{at}(0)=|e\rangle\langle e|$ and

$$\rho_\mathrm{ab}(0)=\rho_\mathrm{a}(0)\otimes\rho_\mathrm{b}(0)=\iint\frac{\mathrm{d}^2\alpha_\mathrm{a}\,\mathrm{d}^2\alpha_\mathrm{b}}{\pi^2}P_\mathrm{a}(\alpha_\mathrm{a})P_\mathrm{b}(\alpha_\mathrm{b})|\alpha_\mathrm{a},\alpha_\mathrm{b}\rangle\langle\alpha_\mathrm{a},\alpha_\mathrm{b}| \tag{10}$$

where $P(\alpha)$ represents the Glauber–Sudarshan quasiprobability distribution for each field, and $|\alpha_a, \alpha_b\rangle \equiv |\alpha_a\rangle \otimes |\alpha_b\rangle$. Consequently, the matrix elements $\boldsymbol{\rho}_{ij}(t)$ can be calculated through the expressions

$$\boldsymbol{\rho}_{11}(t) = \boldsymbol{\mathcal{U}}_{11}(t)\boldsymbol{\rho}_{ab}(0)\boldsymbol{\mathcal{U}}_{11}^{\dagger}(t) \tag{11}$$

$$\boldsymbol{\rho}_{12}(t) = \boldsymbol{\mathcal{U}}_{11}(t)\boldsymbol{\rho}_{ab}(0)\boldsymbol{\mathcal{U}}_{21}^{\dagger}(t) \tag{12}$$

$$\boldsymbol{\rho}_{21}(t) = \boldsymbol{\mathcal{U}}_{21}(t)\boldsymbol{\rho}_{ab}(0)\boldsymbol{\mathcal{U}}_{11}^{\dagger}(t) \tag{13}$$

$$\boldsymbol{\rho}_{22}(t) = \boldsymbol{\mathcal{U}}_{21}(t)\boldsymbol{\rho}_{ab}(0)\boldsymbol{\mathcal{U}}_{21}^{\dagger}(t). \tag{14}$$

Note that $\boldsymbol{\rho}(t)$ describes the exact solution of the Schrödinger equation in the interaction picture with the nonresonant driven-JCM Hamiltonian. Using this solution we can establish analytical expressions for the time evolution of various functions characterizing the quantum state of the cavity field, such as the atomic inversion, the moments associated with the photon-number operator, the Mandel Q parameter, the photon-number distribution and its respective entropy, the variances of the quadrature components and the Wigner function. For simplicity, the initial state of the external quantum field will be fixed in the coherent state throughout this paper ($\boldsymbol{\rho}_b(0) = |\beta\rangle\langle\beta|$), and the initial state of the cavity field will assume two different possibilities: the even- and odd-coherent states [29]. With respect to the Glauber–Sudarshan quasiprobability distribution, these considerations are equivalent to $P_b^{(c)}(\alpha_b) = \pi\delta^{(2)}(\alpha_b - \beta)$ and

$$P_a^{(e)}(\alpha_a) = \frac{\pi}{4}\frac{\exp(|\alpha_a|^2)}{\cosh(|\alpha|^2)}\left[\delta^{(2)}(\alpha_a - \alpha) + \delta^{(2)}(\alpha_a + \alpha) + 2\cosh\left(\alpha\frac{\partial}{\partial\alpha_a} - \alpha^*\frac{\partial}{\partial\alpha_a^*}\right)\delta^{(2)}(\alpha_a)\right]$$

$$P_a^{(o)}(\alpha_a) = \frac{\pi}{4}\frac{\exp(|\alpha_a|^2)}{\sinh(|\alpha|^2)}\left[\delta^{(2)}(\alpha_a - \alpha) + \delta^{(2)}(\alpha_a + \alpha) - 2\cosh\left(\alpha\frac{\partial}{\partial\alpha_a} - \alpha^*\frac{\partial}{\partial\alpha_a^*}\right)\delta^{(2)}(\alpha_a)\right]$$

$\delta^{(2)}(z)$ being the two-dimensional delta function. According to Glauber [28]: 'if the singularities of $P(\alpha)$ are of a type stronger than those of a delta function, e.g., derivatives of a delta function, the field represented will have no classical analogue.' Thus, in the following sections we will investigate the influence of the amplitude of the external field and detuning parameters on the nonclassical behaviour of the cavity field where, in particular, the atomic inversion and the Wigner function should be emphasized.

3. Atomic inversion

The atomic inversion $\mathcal{I}(t) \equiv \mathrm{Tr}[\boldsymbol{\rho}(t)\boldsymbol{\sigma}_z]$ is a quantity of central interest in this section since it is easily accessible in experiments [36]. For the atom–cavity system described in the previous section, this function can be written in an integral form as (see appendix A for calculational details)

$$\mathcal{I}(t) = \iint \frac{d^2\alpha_a\, d^2\alpha_b}{\pi^2} P_a(\alpha_a) P_b(\alpha_b)\, \Xi(\alpha_a, \alpha_b; t) \tag{15}$$

where

$$\Xi(\alpha_a, \alpha_b; t) = 1 - 2\exp(-|\epsilon_a\alpha_a + \epsilon_b\alpha_b|^2)\sum_{n=0}^{\infty}\frac{|\epsilon_a\alpha_a + \epsilon_b\alpha_b|^{2n}}{n!}|G_n(t)|^2.$$

Here, the function $G_n(t) = -\mathrm{i}(\Omega_n/\Delta_n)\sin(\Delta_n t/2)$ is responsible for the time evolution of the atomic inversion, with $\Delta_n^2 = \delta^2 + \Omega_n^2$ and $\Omega_n = 2\kappa_{\mathrm{eff}}\sqrt{n+1}$ the effective Rabi frequency. Note that equation (15) can be obtained for any states of the cavity and external electromagnetic fields. For instance, if one considers both the cavity and external fields in coherent states, the atomic inversion coincides with $\Xi(\alpha, \beta; t)$. This situation was investigated by Dutra *et al* [17] for the atomic excitation probability $P_{\mathrm{e}}(t) = \frac{1}{2}[\mathcal{I}(t) + 1]$ and $\delta = 0$ (resonance condition), where the authors showed that $P_{\mathrm{e}}(t)$ is connected to the Wigner characteristic function of the cavity field, since the conditions: $\kappa_{\mathrm{a}} \gg \kappa_{\mathrm{b}}, \kappa_{\mathrm{b}} t \ll 1$ and $|\beta| \gg (\kappa_{\mathrm{a}}/\kappa_{\mathrm{b}})\kappa_{\mathrm{a}} t$ (intense driving field) are satisfied. Now, if one considers the cavity and external fields in the thermal and coherent states, respectively, the atomic inversion is given by

$$\mathcal{I}_{\mathrm{th}}(t) = 1 - \frac{2}{1+\epsilon_{\mathrm{a}}^2\bar{n}} \exp\left(-\frac{\epsilon_{\mathrm{b}}^2|\beta|^2}{1+\epsilon_{\mathrm{a}}^2\bar{n}}\right) \sum_{n=0}^{\infty} \left(\frac{\epsilon_{\mathrm{a}}^2\bar{n}}{1+\epsilon_{\mathrm{a}}^2\bar{n}}\right)^n L_n\left[-\frac{\epsilon_{\mathrm{b}}^2|\beta|^2}{\epsilon_{\mathrm{a}}^2\bar{n}\left(1+\epsilon_{\mathrm{a}}^2\bar{n}\right)}\right] |G_n(t)|^2. \quad (16)$$

In this expression, $\bar{n}$ is the mean number of thermal photons at time $t = 0$, and $L_n(z)$ corresponds to a Laguerre polynomial. Furthermore, the parameter $\epsilon_{\mathrm{a(b)}}$ represents a scale factor for $\bar{n}(|\beta|)$. It is important to mention that equation (16) corroborates the numerical investigations realized by Li and Gao [21] for the thermal states, and this fact leads us to proceed with the study of atomic inversion for the even- and odd-coherent states.

Let us consider the Glauber–Sudarshan quasiprobability distributions $P_{\mathrm{a}}^{(\mathrm{e})}(\alpha_{\mathrm{a}})$ and $P_{\mathrm{a}}^{(\mathrm{o})}(\alpha_{\mathrm{a}})$ for the even- and odd-coherent states in equation (15), whose integrals in the complex α_{a}- and α_{b}-planes can be evaluated without technical difficulties. In both situations, the atomic inversion is expressed in the compact forms

$$\mathcal{I}_{\mathrm{e}}(t) = 1 - 2\sum_{n=0}^{\infty} \mathfrak{F}_n^{(\mathrm{e})}(\alpha, \beta)|G_n(t)|^2 \quad (17)$$

$$\mathcal{I}_{\mathrm{o}}(t) = 1 - 2\sum_{n=0}^{\infty} \mathfrak{F}_n^{(\mathrm{o})}(\alpha, \beta)|G_n(t)|^2 \quad (18)$$

with $\mathfrak{F}_n^{(\mathrm{e})}(\alpha, \beta)$ and $\mathfrak{F}_n^{(\mathrm{o})}(\alpha, \beta)$ given by

$$\begin{aligned}\mathfrak{F}_n^{(\mathrm{e})}(\alpha, \beta) = \frac{\exp(|\alpha|^2)}{4\cosh(|\alpha|^2)} \Bigg\{ &\exp(-|\epsilon_{\mathrm{a}}\alpha + \epsilon_{\mathrm{b}}\beta|^2)\frac{|\epsilon_{\mathrm{a}}\alpha + \epsilon_{\mathrm{b}}\beta|^{2n}}{n!} + \exp(-|\epsilon_{\mathrm{a}}\alpha - \epsilon_{\mathrm{b}}\beta|^2) \\ &\times \frac{|\epsilon_{\mathrm{a}}\alpha - \epsilon_{\mathrm{b}}\beta|^{2n}}{n!} + 2\exp(-2|\alpha|^2)\mathrm{Re}\Bigg[\exp[(\epsilon_{\mathrm{a}}\alpha + \epsilon_{\mathrm{b}}\beta)(\epsilon_{\mathrm{a}}\alpha - \epsilon_{\mathrm{b}}\beta)^*] \\ &\times \frac{[-(\epsilon_{\mathrm{a}}\alpha + \epsilon_{\mathrm{b}}\beta)(\epsilon_{\mathrm{a}}\alpha - \epsilon_{\mathrm{b}}\beta)^*]^n}{n!}\Bigg]\Bigg\}\end{aligned}$$

$$\begin{aligned}\mathfrak{F}_n^{(\mathrm{o})}(\alpha, \beta) = \frac{\exp(|\alpha|^2)}{4\sinh(|\alpha|^2)} \Bigg\{ &\exp(-|\epsilon_{\mathrm{a}}\alpha + \epsilon_{\mathrm{b}}\beta|^2)\frac{|\epsilon_{\mathrm{a}}\alpha + \epsilon_{\mathrm{b}}\beta|^{2n}}{n!} + \exp(-|\epsilon_{\mathrm{a}}\alpha - \epsilon_{\mathrm{b}}\beta|^2) \\ &\times \frac{|\epsilon_{\mathrm{a}}\alpha - \epsilon_{\mathrm{b}}\beta|^{2n}}{n!} - 2\exp(-2|\alpha|^2)\mathrm{Re}\Bigg[\exp[(\epsilon_{\mathrm{a}}\alpha + \epsilon_{\mathrm{b}}\beta)(\epsilon_{\mathrm{a}}\alpha - \epsilon_{\mathrm{b}}\beta)^*] \\ &\times \frac{[-(\epsilon_{\mathrm{a}}\alpha + \epsilon_{\mathrm{b}}\beta)(\epsilon_{\mathrm{a}}\alpha - \epsilon_{\mathrm{b}}\beta)^*]^n}{n!}\Bigg]\Bigg\}.\end{aligned}$$

Figure 2 shows the plots of $\mathcal{I}_{\mathrm{e}}(t)$ versus $\kappa_{\mathrm{eff}} t$ when the atom–cavity system is resonant (*a*), (*c*) $\delta = 0$ and nonresonant (*b*), (*d*) $\delta = 6\kappa_{\mathrm{eff}}$, for $\epsilon_{\mathrm{a}} = 3/\sqrt{10}$, $\epsilon_{\mathrm{b}} = 1/\sqrt{10}$ and $|\alpha| = 1$ fixed, with two different values of amplitude of the external field: (*a*), (*b*) $|\beta| = 2$ and (*c*), (*d*) $|\beta| = 20$. Since the atom was initially prepared in the excited state, the value of theatomic inversion at the

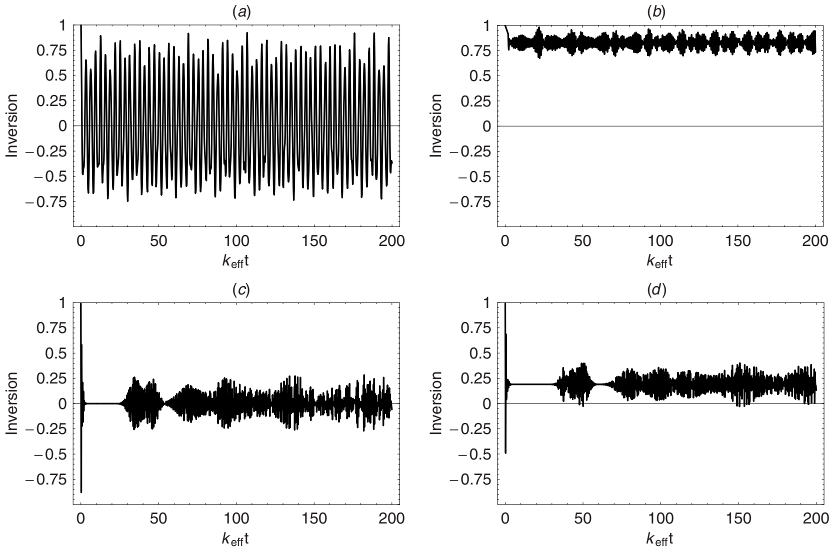

Figure 2. Time evolution of $\mathcal{I}_e(t)$ when the atom is initially prepared in the excited state, and both the cavity and external fields are prepared in the even-coherent and coherent states, respectively. These pictures correspond to (*a*), (*c*) $\delta = 0$ (resonant) and (*b*), (*d*) $\delta = 6\kappa_{\rm eff}$ (nonresonant), for $\epsilon_a = 3/\sqrt{10}$, $\epsilon_b = 1/\sqrt{10}$ and $|\alpha| = 1$ fixed, where two different values of amplitude of the external field were considered: (*a*), (*b*) $|\beta| = 2$ and (*c*), (*d*) $|\beta| = 20$.

time origin is equal to unity in all situations. In figure 2(*a*), we can perceive that $\mathcal{I}_e(t)$ behaves in a fairly irregular manner and the revivals are not well defined (in particular, the revivals are considered as a manifestation of the quantum nature of the electromagnetic field inside the cavity); while in figure 2(*c*), the collapses and revivals appear when the external field is strong. Now, if one analyses figures 2(*b*) and (*d*), then one concludes that the collapses and revivals can be controlled by the detuning between the cavity (external) field and the atomic transition (in particular, the revivals have a regular structure and small amplitude). Similarly, figure 3 shows the plots of $\mathcal{I}_o(t)$ versus $\kappa_{\rm eff} t$ considering the same parameter set used in the previous figure, where we verify that: (i) different structures of collapses and revivals are present and (ii) the effects of the parameters $|\beta|$ and δ on the atomic inversion $\mathcal{I}_o(t)$ are completely analogous to the even coherent states. Góra and Jedrzejek [37] have shown that in the usual JCM with the cavity field prepared initially in a coherent state with a small mean number of photons (i.e., $\langle \mathbf{n} \rangle_c \approx 2$ at time $t = 0$), the atomic inversion displays distinct collapses and revivals provided the atom and the field are slightly detuned, and the *long-time* behaviour of the model presents superstructures such as fractional revivals and super-revivals. In this sense, figures 2(*b*) and 3(*b*) present a *short-period* behaviour with analogous superstructures and this fact is associated with the small mean number of photons used for both the cavity and external fields ($\langle \mathbf{n}_a(0) \rangle_e \approx 0.762$ and $\langle \mathbf{n}_a(0) \rangle_o \approx 1.313$, with $\langle \mathbf{n}_b(0) \rangle_c \approx 4$ fixed), since the detuning is large as compared to the effective coupling constant (e.g., $\delta/2\kappa_{\rm eff} = 3$). Moreover, these superstructures disappear when we consider $\langle \mathbf{n}_b(0) \rangle_c \approx 400$ in figures 2(*d*) and 3(*d*). Summarizing, the amplitude of the external field and the detuning parameter have a strong influence on the structures of collapses and revivals in the JCM with an external quantum field, and this fact leads us to investigate their effects on the nonclassical properties of the cavity field via the Wigner function.

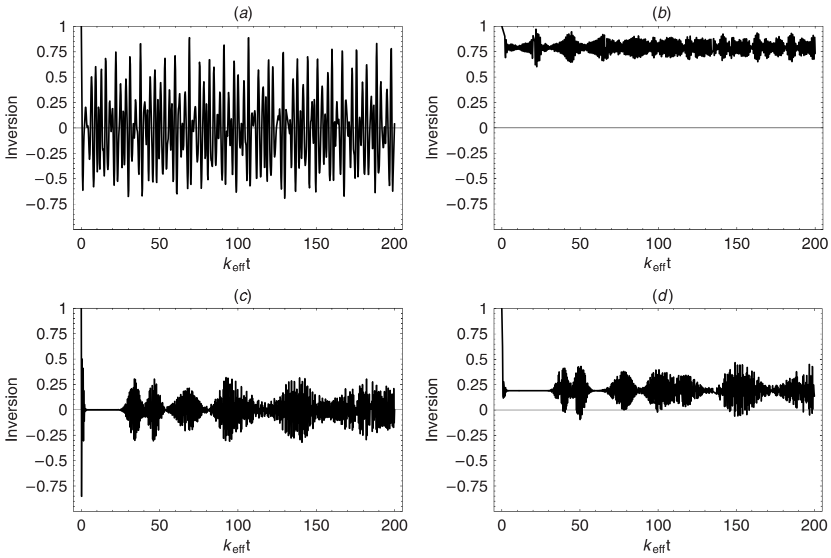

Figure 3. Plots of $\mathcal{I}_{\rm o}(t)$ versus $\kappa_{\rm eff}t \in [0, 200]$ for (*a*), (*c*) $\delta = 0$ (resonant) and (*b*), (*d*) $\delta = 6\kappa_{\rm eff}$ (nonresonant), with $\epsilon_{\rm a} = 3/\sqrt{10}$, $\epsilon_{\rm b} = 1/\sqrt{10}$ and $|\alpha| = 1$ fixed. In both situations, different values of the amplitude of the external field were considered, i.e., (*a*), (*b*) $|\beta| = 2$ and (*c*), (*d*) $|\beta| = 20$.

4. The Wigner function

In many recent textbooks on quantum optics [35], the Wigner function is generally defined in terms of an auxiliary function (also denominated as the Wigner characteristic function) which describes the symmetric ordering of creation and annihilation operators of the electromagnetic field, i.e., $\chi(\xi) \equiv \mathrm{Tr}[\boldsymbol{\rho}\mathbf{D}(\xi)]$ with $\mathbf{D}(\xi) = \exp(\xi\mathbf{a}^{\dagger} - \xi^{*}\mathbf{a})$ being the displacement operator. The connection between both functions is established by means of a two-dimensional Fourier transform as follows:

$$W(\gamma) = \int \frac{\mathrm{d}^2\xi}{\pi} \exp(\gamma\xi^{*} - \gamma^{*}\xi)\chi(\xi). \tag{19}$$

Thus, if one considers the cavity field in the framework of the JCM with an external quantum field, its Wigner characteristic function can be defined in a similar form to atomic inversion,

$$\chi(\xi; t) = \iint \frac{\mathrm{d}^2\alpha_{\rm a}\,\mathrm{d}^2\alpha_{\rm b}}{\pi^2} P_{\rm a}(\alpha_{\rm a}) P_{\rm b}(\alpha_{\rm b}) \tilde{\rm K}_{\xi}(\alpha_{\rm a}, \alpha_{\rm b}; t) \tag{20}$$

with $\tilde{\rm K}_{\xi}(\alpha_{\rm a}, \alpha_{\rm b}; t)$ given by

$$\tilde{\rm K}_{\xi}(\alpha_{\rm a}, \alpha_{\rm b}; t) = \langle\alpha_{\rm a}, \alpha_{\rm b}|\boldsymbol{\mathcal{U}}_{11}^{\dagger}(t)\mathbf{D}_{\rm a}(\xi)\boldsymbol{\mathcal{U}}_{11}(t)|\alpha_{\rm a}, \alpha_{\rm b}\rangle + \langle\alpha_{\rm a}, \alpha_{\rm b}|\boldsymbol{\mathcal{U}}_{21}^{\dagger}(t)\mathbf{D}_{\rm a}(\xi)\boldsymbol{\mathcal{U}}_{21}(t)|\alpha_{\rm a}, \alpha_{\rm b}\rangle.$$

Here, the displacement operator $\mathbf{D}_{\rm a}(\xi)$ is associated with the cavity field. Now, substituting $\chi(\xi; t)$ into equation (19), the expression for the Wigner function is promptly obtained,

$$W_{\rm a}(\gamma; t) = \iint \frac{\mathrm{d}^2\alpha_{\rm a}\,\mathrm{d}^2\alpha_{\rm b}}{\pi^2} P_{\rm a}(\alpha_{\rm a}) P_{\rm b}(\alpha_{\rm b}) {\rm K}_{\gamma}(\alpha_{\rm a}, \alpha_{\rm b}; t) \tag{21}$$

where the label γ corresponds to representation in the complex phase space and

$${\rm K}_{\gamma}(\alpha_{\rm a}, \alpha_{\rm b}; t) = \int \frac{\mathrm{d}^2\xi}{\pi} \exp(\gamma\xi^{*} - \gamma^{*}\xi)\, \tilde{\rm K}_{\xi}(\alpha_{\rm a}, \alpha_{\rm b}; t). \tag{22}$$

The functions $\tilde{\mathrm{K}}_{\xi}(\alpha_{\mathrm{a}}, \alpha_{\mathrm{b}}; t)$ and $\mathrm{K}_{\gamma}(\alpha_{\mathrm{a}}, \alpha_{\mathrm{b}}; t)$ were derived with details in appendix B. In particular, when $t = 0$ the function $\mathrm{K}_{\gamma}(\alpha_{\mathrm{a}}, \alpha_{\mathrm{b}}; 0) = 2\exp(-2|\gamma - \alpha_{\mathrm{a}}|^2)$ does not depend on variables associated with the external field and this fact leads one to write the initial Wigner function as

$$W_{\mathrm{a}}(\gamma; 0) = 2\int \frac{\mathrm{d}^2\alpha_{\mathrm{a}}}{\pi} \exp(-2|\gamma - \alpha_{\mathrm{a}}|^2) P_{\mathrm{a}}(\alpha_{\mathrm{a}}).$$

This expression represents a Gaussian smoothing process of the integrand $P_{\mathrm{a}}(\alpha_{\mathrm{a}})$ such that $W_{\mathrm{a}}(\gamma; 0)$ is a well-defined function in the phase space $p = \sqrt{2}\,\mathrm{Im}(\gamma)$ and $q = \sqrt{2}\,\mathrm{Re}(\gamma)$. When $t > 0$, the function $\mathrm{K}_{\gamma}(\alpha_{\mathrm{a}}, \alpha_{\mathrm{b}}; t)$ is responsible for the entanglement between the atom and both the cavity and external fields (here represented by the Glauber–Sudarshan quasiprobability distributions $P_{\mathrm{a}}(\alpha_{\mathrm{a}})$ and $P_{\mathrm{b}}(\alpha_{\mathrm{b}})$, respectively) since the complex variables α_{a} and α_{b} are completely correlated. Furthermore, it is important to mention that $\chi(\xi; t)$ and $W_{\mathrm{a}}(\gamma; t)$ can be evaluated for any states of the cavity and external fields (a similar condition was established for atomic inversion) without restrictions on the different interaction times, and the expressions obtained analytically from this procedure generalize the results previously discussed in the literature [15, 17].

For instance, let us consider the Glauber–Sudarshan quasiprobability distributions for even- and odd-coherent states in equation (21). After the integrations in the complex α_{a}- and α_{b}-planes, we get

$$W_{\mathrm{a}}^{(\mathrm{e})}(\gamma; t) = \frac{\exp(|\alpha|^2)}{4\cosh(|\alpha|^2)}[\mathrm{K}_{\gamma}(\alpha, \beta; t) + \mathrm{K}_{\gamma}(-\alpha, \beta; t) + \exp(-2|\alpha|^2)\mathbb{K}_{\gamma}(\alpha, \beta; t)] \tag{23}$$

and

$$W_{\mathrm{a}}^{(\mathrm{o})}(\gamma; t) = \frac{\exp(|\alpha|^2)}{4\sinh(|\alpha|^2)}[\mathrm{K}_{\gamma}(\alpha, \beta; t) + \mathrm{K}_{\gamma}(-\alpha, \beta; t) - \exp(-2|\alpha|^2)\mathbb{K}_{\gamma}(\alpha, \beta; t)] \tag{24}$$

where

$$\mathbb{K}_{\gamma}(\alpha, \beta; t) = 2\exp(|\alpha|^2)\cosh\left(\alpha\frac{\partial}{\partial\alpha_{\mathrm{a}}} - \alpha^*\frac{\partial}{\partial\alpha_{\mathrm{a}}^*}\right)[\exp(|\alpha_{\mathrm{a}}|^2)\mathrm{K}_{\gamma}(\alpha_{\mathrm{a}}, \beta; t)]\bigg|_{\alpha_{\mathrm{a}}=0}. \tag{25}$$

Note that at time $t = 0$, the function $\exp(-2|\alpha|^2)\mathbb{K}_{\gamma}(\alpha, \beta; 0) = 4\exp(-2|\gamma|^2)\cos[4\,\mathrm{Im}(\gamma\alpha^*)]$ leads us to recover well-known expressions in the literature [29]:

$$W_{\mathrm{a}}^{(\mathrm{e})}(\gamma; 0) = \frac{\exp(|\alpha|^2)}{\cosh(|\alpha|^2)}\exp(-2|\gamma|^2)\{\exp(-2|\alpha|^2)\cosh[4\,\mathrm{Re}(\gamma\alpha^*)] + \cos[4\,\mathrm{Im}(\gamma\alpha^*)]\}$$

and

$$W_{\mathrm{a}}^{(\mathrm{o})}(\gamma; 0) = \frac{\exp(|\alpha|^2)}{\sinh(|\alpha|^2)}\exp(-2|\gamma|^2)\{\exp(-2|\alpha|^2)\cosh[4\,\mathrm{Re}(\gamma\alpha^*)] - \cos[4\,\mathrm{Im}(\gamma\alpha^*)]\}.$$

Figures 4 and 5 show the three-dimensional plots of $W_{\mathrm{a}}^{(\mathrm{e})}(\gamma; t)$ and $W_{\mathrm{a}}^{(\mathrm{o})}(\gamma; t)$ versus $p = \sqrt{2}\,\mathrm{Im}(\gamma)$ and $q = \sqrt{2}\,\mathrm{Re}(\gamma)$, respectively, for the atom–cavity system resonant (*a*), (*b*) $\delta = 0$ and nonresonant (*c*), (*d*) $\delta = 10\kappa_{\mathrm{eff}}$. In both simulations, we consider $|\alpha| = 1$ and two different values of amplitude of the external quantum field: (*a*), (*c*) $|\beta| = 2$ and (*b*), (*d*) $|\beta| = 5$. Furthermore, we also fix the parameter $\kappa_{\mathrm{eff}}t = 100$ which permits us to obtain a partial view of entanglement in the tripartite system. A first analysis of these pictures shows, via the Wigner function, that the entanglement is sensitive to variations of the experimental parameters $|\beta|$ and δ (this fact corroborates the previous results obtained for atomic inversion); the detuning parameter being responsible for the entanglement degree between the components involved in the system, since both the external and cavity fields are in resonance. In this sense, although figures 4(*c*)–(*d*) and 5(*c*)–(*d*) have similar structures,

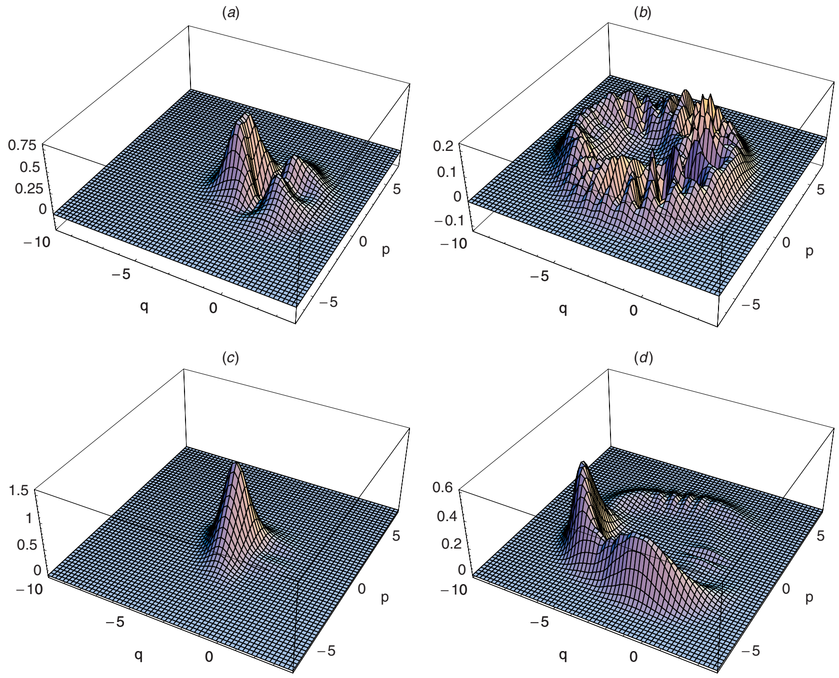

Figure 4. Plots of $W_a^{(e)}(\gamma;t)$ versus $p \in [-7, 7]$ and $q \in [-10, 4]$ for the atom–cavity system with two different values of detuning: (*a*), (*b*) $\delta = 0$ (resonant) and (*c*), (*d*) $\delta = 10\kappa_{\mathrm{eff}}$ (nonresonant), where the parameters $|\alpha| = 1$ ($\langle \mathbf{n}_a\rangle_e \approx 0.762$) and $\kappa_{\mathrm{eff}} t = 100$ were fixed in the present simulation. In both situations, the condition $\kappa_{a(b)} = \kappa$ was established and the values of amplitude of the external quantum field (*a*), (*c*) $|\beta| = 2$ ($\langle \mathbf{n}_b\rangle_c = 4$) and (*b*), (*d*) $|\beta| = 5$ ($\langle \mathbf{n}_b\rangle_c = 25$) were considered.

there are subtle differences between them: $W_a^{(o)}(\gamma;t)$ assumes negative values due to initial sub-Poissonian photon statistics of the cavity field, while $W_a^{(e)}(\gamma;t)$ is strictly positive, since the even-coherent state has super-Poissonian photon statistics for any initial value of $\langle \mathbf{n}_a\rangle_e$ (see [29] for more details). In addition, the increase of $|\beta|$ in figures 4(*b*), (*d*) and 5(*b*), (*d*) shows an interesting effect on the Wigner functions: the correlation patterns between the states of the tripartite system turn out to be more evident, and this effect modifies the shapes of $W_a^{(e)}(\gamma;t)$ and $W_a^{(o)}(\gamma;t)$. A similar analysis can also be applied if one considers both the external and cavity fields in the coherent states (see appendix B).

To conclude this section, we will determine the marginal probability distribution functions $|\psi_a(q;t)|^2$ and $|\varphi_a(p;t)|^2$ through the direct integration of equation (21) over the variables p or q, i.e.,

$$|\psi_a(q;t)|^2 = \iint \frac{d^2\alpha_a\, d^2\alpha_b}{\pi^2} P_a(\alpha_a) P_b(\alpha_b) \mathrm{U}_q(\alpha_a, \alpha_b; t) \tag{26}$$

$$|\varphi_a(p;t)|^2 = \iint \frac{d^2\alpha_a\, d^2\alpha_b}{\pi^2} P_a(\alpha_a) P_b(\alpha_b) \mathrm{V}_p(\alpha_a, \alpha_b; t) \tag{27}$$

with

$$\mathrm{U}_q(\alpha_a, \alpha_b; t) = \int_{-\infty}^{\infty} \frac{dp}{\sqrt{2\pi}} \mathrm{K}_\gamma(\alpha_a, \alpha_b; t)$$

and

$$\mathrm{V}_p(\alpha_a, \alpha_b; t) = \int_{-\infty}^{\infty} \frac{dq}{\sqrt{2\pi}} \mathrm{K}_\gamma(\alpha_a, \alpha_b; t).$$

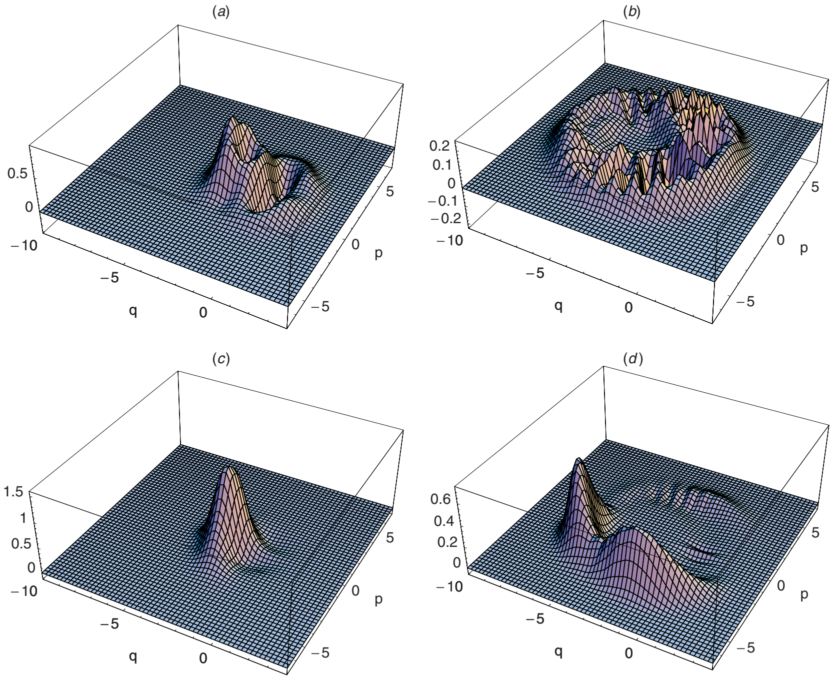

Figure 5. The Wigner function $W_{\rm a}^{({\rm o})}(\gamma;t)$ is plotted assuming the same set of parameters established in the previous figure for the detuning frequency and amplitude of the external field, with $|\alpha| = 1$ ($\langle \mathbf{n}_{\rm a}\rangle_{\rm o} \approx 1.313$) and $\kappa_{\rm eff} t = 100$ fixed. Note that the entanglement is maximum when $\delta = 0$ (resonant regime) and minimum for $\delta = 10\kappa_{\rm eff}$ (nonresonant regime).

For this purpose, let us initially introduce the complex function

$$\mathcal{H}_{\mu}^{(m,m')}(\alpha_{\rm a},\alpha_{\rm b}) = \sum_{k=0}^{\{m,m'\}} (2k)!!\,L_k^{(m-k)}(0)L_k^{(m'-k)}(0)\left[\frac{\epsilon_{\rm a}^3\alpha_{\rm a}+\epsilon_{\rm b}^3\alpha_{\rm b}}{\epsilon_{\rm a}\epsilon_{\rm b}(\epsilon_{\rm b}\alpha_{\rm a}+\epsilon_{\rm a}\alpha_{\rm b})}\right]^k \times H_{m-k}\left(\mu-\frac{\nu_{\rm a}+\nu_{\rm b}^*}{\sqrt{2}}\right)H_{m'-k}\left(\mu-\frac{\nu_{\rm a}+\nu_{\rm b}^*}{\sqrt{2}}\right)$$

where $H_n(z)$ is the Hermite polynomial, $\nu_{\rm a(b)} = \epsilon_{\rm a(b)}(\epsilon_{\rm a(b)}\alpha_{\rm a} - \epsilon_{\rm b(a)}\alpha_{\rm b})$ and $\{m, m'\}$ stands for the minor of m and m'. In addition, we define the auxiliary functions

$$\Upsilon_{\mu}^{(m,m')}(\alpha_{\rm a},\alpha_{\rm b};t) = \mathcal{H}_{\mu}^{(m,m')}(\alpha_{\rm a},\alpha_{\rm b})F_m(t)F_{m'}^*(t) + \frac{\epsilon_{\rm a}\epsilon_{\rm b}}{2}\frac{\epsilon_{\rm b}\alpha_{\rm a}+\epsilon_{\rm a}\alpha_{\rm b}}{\epsilon_{\rm a}\alpha_{\rm a}+\epsilon_{\rm b}\alpha_{\rm b}}\mathcal{H}_{\mu}^{(m+1,m'+1)}(\alpha_{\rm a},\alpha_{\rm b})\frac{G_m(t)G_{m'}^*(t)}{\sqrt{(m+1)(m'+1)}}$$

and

$$\mathcal{A}_{\mu}^{(m,m')}(\alpha_{\rm a},\alpha_{\rm b}) = \sqrt{2}\,\exp\left[-\left(\mu-\frac{\nu_{\rm a}+\nu_{\rm b}^*}{\sqrt{2}}\right)^2 - |\epsilon_{\rm a}\alpha_{\rm a}+\epsilon_{\rm b}\alpha_{\rm b}|^2\right] \times \frac{[\sqrt{2}\epsilon_{\rm b}(\epsilon_{\rm b}\alpha_{\rm a}+\epsilon_{\rm a}\alpha_{\rm b})]^m[\sqrt{2}\epsilon_{\rm a}(\epsilon_{\rm a}\alpha_{\rm a}+\epsilon_{\rm b}\alpha_{\rm b})^*]^{m'}}{(2m)!!(2m')!!}$$

which permit us to express the integrands $\mathrm{U}_q(\alpha_\mathrm{a}, \alpha_\mathrm{b}; t)$ and $\mathrm{V}_p(\alpha_\mathrm{a}, \alpha_\mathrm{b}; t)$ in compact form as follows:

$$\mathrm{U}_q(\alpha_\mathrm{a}, \alpha_\mathrm{b}; t) = \sum_{m,m'=0}^{\infty} \mathcal{A}_q^{(m,m')}(\alpha_\mathrm{a}, \alpha_\mathrm{b}) \mathrm{Y}_q^{(m,m')}(\alpha_\mathrm{a}, \alpha_\mathrm{b}; t) \tag{28}$$

$$\mathrm{V}_p(\alpha_\mathrm{a}, \alpha_\mathrm{b}; t) = \sum_{m,m'=0}^{\infty} \mathcal{A}_p^{(m,m')}(-\mathrm{i}\alpha_\mathrm{a}, -\mathrm{i}\alpha_\mathrm{b}) \mathrm{Y}_p^{(m,m')}(-\mathrm{i}\alpha_\mathrm{a}, -\mathrm{i}\alpha_\mathrm{b}; t). \tag{29}$$

Consequently, the connection between equations (28) and (29) can be promptly established through the mathematical relations $\mathrm{U}_q(\alpha_\mathrm{a}, \alpha_\mathrm{b}; t) = \mathrm{V}_q(\mathrm{i}\alpha_\mathrm{a}, \mathrm{i}\alpha_\mathrm{b}; t)$ and $\mathrm{V}_p(\alpha_\mathrm{a}, \alpha_\mathrm{b}; t) = \mathrm{U}_p(-\mathrm{i}\alpha_\mathrm{a}, -\mathrm{i}\alpha_\mathrm{b}; t)$.

In analogy to the Wigner function, the marginal probability distribution functions do not depend on the external field at time $t = 0$, since their expressions are reduced to

$$|\psi_\mathrm{a}(q; 0)|^2 = \sqrt{2} \int \frac{\mathrm{d}^2\alpha_\mathrm{a}}{\pi} \exp\{-[q - \sqrt{2}\,\mathrm{Re}(\alpha_\mathrm{a})]^2\} P_\mathrm{a}(\alpha_\mathrm{a})$$
$$|\varphi_\mathrm{a}(p; 0)|^2 = \sqrt{2} \int \frac{\mathrm{d}^2\alpha_\mathrm{a}}{\pi} \exp\{-[p - \sqrt{2}\,\mathrm{Im}(\alpha_\mathrm{a})]^2\} P_\mathrm{a}(\alpha_\mathrm{a}).$$

Now, if one considers both the external and cavity fields in the coherent states, then one gets $|\psi_\mathrm{a}(q; t)|^2 = \mathrm{U}_q(\alpha, \beta; t)$ and $|\varphi_\mathrm{a}(p; t)|^2 = \mathrm{V}_p(\alpha, \beta; t)$. In particular, this example shows that the marginal distributions represent an important additional tool in the qualitative study of entanglement, since the variables α and β are completely correlated.

5. Summary and conclusions

In this paper we have applied the decomposition formula for $su(2)$ Lie algebra on the Jaynes–Cummings model with an external quantum field in order to calculate, for instance, the exact expressions for atomic inversion and the Wigner function when the atom is initially prepared in the excited state. In fact, adopting the diagonal representation of coherent states, we have shown that these expressions can be written in the integral form, with their integrands presenting a common term which describes the product of the Glauber–Sudarshan quasiprobability distribution functions for each field, and a kernel responsible for the entanglement. It is important to mention that the mathematical procedure developed here does not present any restrictions on the states of the cavity and external electromagnetic fields. Following, to illustrate these results we have fixed the external field in the coherent state and assumed two different possibilities for the cavity field (i.e., the even- and odd-coherent states). In this way, we have verified that the amplitude of the external field and detuning parameter (i) perform a strong influence on the structures of collapses and revivals in the atomic inversion, (ii) control the entanglement degree in the tripartite system; and consequently, (iii) modify the shape of $W_\mathrm{a}(\gamma; t)$ since the correlation patterns between the states of the tripartite system turn out to be more evident through the Wigner function. In addition, the formalism employed in the calculation of atomic inversion and Wigner function opens new possibilities of future investigations in similar physical systems (e.g., [26, 27]); or in the study of dissipative composite systems, where the decoherence effect has a central role in the quantum information processing. These considerations are under current research and will be published elsewhere. Summarizing, the work reported here is clearly the product of considerable effort and represents an original contribution to the wider field of entangled-state engineering with emphasis on quantum computation and related topics.

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Appendix A. The integral form of the atomic inversion

With the help of the definition established in section 3 for atomic inversion and the cyclic invariance property of the trace operation, we get

$$\begin{aligned}\mathcal{I}(t) &= \mathrm{Tr}_{\mathrm{ab}}\left\{\rho_{\mathrm{ab}}(0)\left[\mathcal{U}_{11}^{\dagger}(t)\mathcal{U}_{11}(t) - \mathcal{U}_{21}^{\dagger}(t)\mathcal{U}_{21}(t)\right]\right\} \\ &= \mathrm{Tr}_{\mathrm{ab}}\left\{\rho_{\mathrm{ab}}(0)\left[\cos(2t\sqrt{\beta_{\mathrm{A}}}) + \frac{\delta^2}{2}\frac{\sin^2(t\sqrt{\beta_{\mathrm{A}}})}{\beta_{\mathrm{A}}}\right]\right\}. \end{aligned} \tag{A.1}$$

Employing the diagonal representation of $\rho_{\mathrm{ab}}(0)$ in the coherent-state basis in the second equality of equation (A.1), the integral form of the atomic inversion can be promptly obtained, i.e.,

$$\mathcal{I}(t) = \iint \frac{\mathrm{d}^2\alpha_{\mathrm{a}}\,\mathrm{d}^2\alpha_{\mathrm{b}}}{\pi^2} P_{\mathrm{a}}(\alpha_{\mathrm{a}})P_{\mathrm{b}}(\alpha_{\mathrm{b}})\,\Xi(\alpha_{\mathrm{a}},\alpha_{\mathrm{b}};t) \tag{A.2}$$

where

$$\Xi(\alpha_{\mathrm{a}},\alpha_{\mathrm{b}};t) = \langle\alpha_{\mathrm{a}},\alpha_{\mathrm{b}}|\cos(2t\sqrt{\beta_{\mathrm{A}}})|\alpha_{\mathrm{a}},\alpha_{\mathrm{b}}\rangle + \frac{\delta^2}{2}\langle\alpha_{\mathrm{a}},\alpha_{\mathrm{b}}|\frac{\sin^2(t\sqrt{\beta_{\mathrm{A}}})}{\beta_{\mathrm{A}}}|\alpha_{\mathrm{a}},\alpha_{\mathrm{b}}\rangle. \tag{A.3}$$

However, the effectiveness of the integral form (A.2) is connected with the determination of an analytical expression for equation (A.3).

To calculate the function $\Xi(\alpha_{\mathrm{a}},\alpha_{\mathrm{b}};t)$, first we expand the operators $\cos(2t\sqrt{\beta_{\mathrm{A}}})$ and $\sin^2(t\sqrt{\beta_{\mathrm{A}}})/\beta_{\mathrm{A}}$ in a power series as follows:

$$\cos(2t\sqrt{\beta_{\mathrm{A}}}) = \sum_{k=0}^{\infty}\frac{(-1)^k}{(2k)!}(\sqrt{2}\kappa_{\mathrm{eff}}t)^{2k}\frac{\mathrm{d}^k}{\mathrm{d}x^k}\exp\left(2x\left(1+\delta^2/4\kappa_{\mathrm{eff}}^2\right)\right)\exp(x\mathbf{D})\exp(x\mathbf{S})|_{x=0}$$

and

$$\begin{aligned}\frac{\sin^2(t\sqrt{\beta_{\mathrm{A}}})}{\beta_{\mathrm{A}}} &= \frac{1}{\kappa_{\mathrm{eff}}^2}\sum_{k=0}^{\infty}\frac{(-1)^k}{[2(k+1)]!}(\sqrt{2}\kappa_{\mathrm{eff}}t)^{2(k+1)}\frac{\mathrm{d}^k}{\mathrm{d}x^k} \\ &\quad\times\exp\left(2x\left(1+\delta^2/4\kappa_{\mathrm{eff}}^2\right)\right)\exp(x\mathbf{D})\exp(x\mathbf{S})|_{x=0}.\end{aligned}$$

Secondly, we apply the antinormal-order decomposition formula for $su(2)$ Lie algebra on the operator $\exp(x\mathbf{D})$, which leads us to obtain [38–40]

$$\exp(x\mathbf{D}) = \exp(B_{+}\mathbf{K}_{-})\exp(B_{+}B_0\mathbf{K}_{+})\exp((\ln B_0)\mathbf{K}_0)$$

where

$$B_{+} = \frac{2\epsilon_{\mathrm{a}}\epsilon_{\mathrm{b}}\sinh x}{\cosh x + \left(\epsilon_{\mathrm{a}}^2 - \epsilon_{\mathrm{b}}^2\right)\sinh x} \qquad \text{and} \qquad B_0 = \left[\cosh x + \left(\epsilon_{\mathrm{a}}^2 - \epsilon_{\mathrm{b}}^2\right)\sinh x\right]^2.$$

After lengthy calculations, the analytical expressions for the mean values

$$\langle\alpha_{\mathrm{a}},\alpha_{\mathrm{b}}|\cos(2t\sqrt{\beta_{\mathrm{A}}})|\alpha_{\mathrm{a}},\alpha_{\mathrm{b}}\rangle = \exp(-|\epsilon_{\mathrm{a}}\alpha_{\mathrm{a}}+\epsilon_{\mathrm{b}}\alpha_{\mathrm{b}}|^2)\sum_{n=0}^{\infty}\frac{|\epsilon_{\mathrm{a}}\alpha_{\mathrm{a}}+\epsilon_{\mathrm{b}}\alpha_{\mathrm{b}}|^{2n}}{n!}\cos(\Delta_n t) \tag{A.4}$$

and

$$\langle\alpha_{\rm a},\alpha_{\rm b}|\frac{\sin^2(t\sqrt{\beta_{\rm A}})}{\beta_{\rm A}}|\alpha_{\rm a},\alpha_{\rm b}\rangle = \exp(-|\epsilon_{\rm a}\alpha_{\rm a}+\epsilon_{\rm b}\alpha_{\rm b}|^2)\sum_{n=0}^{\infty}\frac{|\epsilon_{\rm a}\alpha_{\rm a}+\epsilon_{\rm b}\alpha_{\rm b}|^{2n}}{n!}\frac{\sin^2(\Delta_n t/2)}{(\Delta_n/2)^2} \quad (A.5)$$

are determined, with $\Delta_n^2 = \delta^2 + \Omega_n^2$, and $\Omega_n = 2\kappa_{\rm eff}\sqrt{n+1}$ being the effective Rabi frequency. Now, substituting these results into equation (A.3), we obtain

$$\Xi(\alpha_{\rm a},\alpha_{\rm b};t) = 1 - 2\exp(-|\epsilon_{\rm a}\alpha_{\rm a}+\epsilon_{\rm b}\alpha_{\rm b}|^2)\sum_{n=0}^{\infty}\frac{|\epsilon_{\rm a}\alpha_{\rm a}+\epsilon_{\rm b}\alpha_{\rm b}|^{2n}}{n!}|G_n(t)|^2 \quad (A.6)$$

where $G_n(t) = -{\rm i}(\Omega_n/\Delta_n)\sin(\Delta_n t/2)$. Consequently, with the determination of the analytical expression for $\Xi(\alpha_{\rm a},\alpha_{\rm b};t)$, the effectiveness of the integral form (A.2) is guaranteed.

Appendix B. Calculational details of the Wigner function

Initially, we will derive the mean values

$$\langle\alpha_{\rm a},\alpha_{\rm b}|\boldsymbol{\mathcal{U}}_{11}^{\dagger}(t)\mathbf{D}_{\rm a}(\xi)\boldsymbol{\mathcal{U}}_{11}(t)|\alpha_{\rm a},\alpha_{\rm b}\rangle = \iint\frac{{\rm d}^2\beta_{\rm a}\,{\rm d}^2\beta_{\rm b}}{\pi^2}\langle\alpha_{\rm a},\alpha_{\rm b}|\boldsymbol{\mathcal{U}}_{11}^{\dagger}(t)\mathbf{D}_{\rm a}(\xi)|\beta_{\rm a},\beta_{\rm b}\rangle\langle\beta_{\rm a},\beta_{\rm b}|\boldsymbol{\mathcal{U}}_{11}(t)|\alpha_{\rm a},\alpha_{\rm b}\rangle \quad (B.1)$$

and

$$\langle\alpha_{\rm a},\alpha_{\rm b}|\boldsymbol{\mathcal{U}}_{21}^{\dagger}(t)\mathbf{D}_{\rm a}(\xi)\boldsymbol{\mathcal{U}}_{21}(t)|\alpha_{\rm a},\alpha_{\rm b}\rangle = \iint\frac{{\rm d}^2\beta_{\rm a}\,{\rm d}^2\beta_{\rm b}}{\pi^2}\langle\alpha_{\rm a},\alpha_{\rm b}|\boldsymbol{\mathcal{U}}_{21}^{\dagger}(t)\mathbf{D}_{\rm a}(\xi)|\beta_{\rm a},\beta_{\rm b}\rangle\langle\beta_{\rm a},\beta_{\rm b}|\boldsymbol{\mathcal{U}}_{21}(t)|\alpha_{\rm a},\alpha_{\rm b}\rangle \quad (B.2)$$

by means of integrations in the complex variables $\beta_{\rm a}$ and $\beta_{\rm b}$. Thus, let us substitute into equations (B.1) and (B.2) the auxiliary mean values

$$\langle\alpha_{\rm a},\alpha_{\rm b}|\boldsymbol{\mathcal{U}}_{11}^{\dagger}(t)\mathbf{D}_{\rm a}(\xi)|\beta_{\rm a},\beta_{\rm b}\rangle = \exp\left[\tfrac{1}{2}(\xi\beta_{\rm a}^* - \xi^*\beta_{\rm a})\right](\langle\beta_{\rm a}+\xi,\beta_{\rm b}|\boldsymbol{\mathcal{U}}_{11}(t)|\alpha_{\rm a},\alpha_{\rm b}\rangle)^*$$

$$\langle\alpha_{\rm a},\alpha_{\rm b}|\boldsymbol{\mathcal{U}}_{21}^{\dagger}(t)\mathbf{D}_{\rm a}(\xi)|\beta_{\rm a},\beta_{\rm b}\rangle = \exp\left[\tfrac{1}{2}(\xi\beta_{\rm a}^* - \xi^*\beta_{\rm a})\right](\langle\beta_{\rm a}+\xi,\beta_{\rm b}|\boldsymbol{\mathcal{U}}_{21}(t)|\alpha_{\rm a},\alpha_{\rm b}\rangle)^*$$

$$\langle\beta_{\rm a},\beta_{\rm b}|\boldsymbol{\mathcal{U}}_{11}(t)|\alpha_{\rm a},\alpha_{\rm b}\rangle = \sum_{m=0}^{\infty}F_m(t)\Lambda_m(\alpha_{\rm a},\alpha_{\rm b},\beta_{\rm a},\beta_{\rm b})$$

$$\langle\beta_{\rm a},\beta_{\rm b}|\boldsymbol{\mathcal{U}}_{21}(t)|\alpha_{\rm a},\alpha_{\rm b}\rangle = (\epsilon_{\rm a}\beta_{\rm a}+\epsilon_{\rm b}\beta_{\rm b})^*\sum_{m=0}^{\infty}\frac{G_m(t)}{\sqrt{m+1}}\Lambda_m(\alpha_{\rm a},\alpha_{\rm b},\beta_{\rm a},\beta_{\rm b})$$

with

$$\Lambda_m(\alpha_{\rm a},\alpha_{\rm b},\beta_{\rm a},\beta_{\rm b}) = \exp\left[-\tfrac{1}{2}(|\alpha_{\rm a}|^2+|\alpha_{\rm b}|^2+|\beta_{\rm a}|^2+|\beta_{\rm b}|^2) + (\epsilon_{\rm b}\alpha_{\rm a}-\epsilon_{\rm a}\alpha_{\rm b})(\epsilon_{\rm b}\beta_{\rm a}-\epsilon_{\rm a}\beta_{\rm b})^*\right]\frac{[(\epsilon_{\rm a}\alpha_{\rm a}+\epsilon_{\rm b}\alpha_{\rm b})(\epsilon_{\rm a}\beta_{\rm a}+\epsilon_{\rm b}\beta_{\rm b})^*]^m}{m!}$$

and $F_m(t) = \cos(\Delta_m t/2) - {\rm i}(\delta/\Delta_m)\sin(\Delta_m t/2)$ (the function $G_m(t)$ was previously defined in appendix A). Then, carrying out the integrations in the variables $\beta_{\rm a}$ and $\beta_{\rm b}$, we get

$$\langle\alpha_{\rm a},\alpha_{\rm b}|\boldsymbol{\mathcal{U}}_{11}^{\dagger}(t)\mathbf{D}_{\rm a}(\xi)\boldsymbol{\mathcal{U}}_{11}(t)|\alpha_{\rm a},\alpha_{\rm b}\rangle = \sum_{m,m'=0}^{\infty}\Upsilon_{\xi}^{(m,m')}(\alpha_{\rm a},\alpha_{\rm b})\mathrm{I}_{\xi}^{(m,m')}(\alpha_{\rm a},\alpha_{\rm b};t) \quad (B.3)$$

and

$$\langle\alpha_{\rm a},\alpha_{\rm b}|\boldsymbol{\mathcal{U}}_{21}^{\dagger}(t)\mathbf{D}_{\rm a}(\xi)\boldsymbol{\mathcal{U}}_{21}(t)|\alpha_{\rm a},\alpha_{\rm b}\rangle = \sum_{m,m'=0}^{\infty}\Upsilon_{\xi}^{(m,m')}(\alpha_{\rm a},\alpha_{\rm b})\mathrm{J}_{\xi}^{(m,m')}(\alpha_{\rm a},\alpha_{\rm b};t) \quad (B.4)$$

where

$$\Upsilon_\xi^{(m,m')}(\alpha_a,\alpha_b) = \exp\left[-\frac{|\xi|^2}{2} - |\epsilon_a\alpha_a + \epsilon_b\alpha_b|^2 + \epsilon_b(\epsilon_b\alpha_a - \epsilon_a\alpha_b)^*\xi - \epsilon_a(\epsilon_a\alpha_a - \epsilon_b\alpha_b)\xi^*\right] \times [\epsilon_a(\epsilon_a\alpha_a + \epsilon_b\alpha_b)^*\xi]^{m'-m}\frac{|\epsilon_a\alpha_a + \epsilon_b\alpha_b|^{2m}}{m'!}$$

$$\mathrm{I}_\xi^{(m,m')}(\alpha_a,\alpha_b;t) = L_m^{(m'-m)}\left[\epsilon_a\epsilon_b\frac{\epsilon_b\alpha_a + \epsilon_a\alpha_b}{\epsilon_a\alpha_a + \epsilon_b\alpha_b}|\xi|^2\right]F_m(t)F_{m'}^*(t)$$

$$\mathrm{J}_\xi^{(m,m')}(\alpha_a,\alpha_b;t) = \sqrt{\frac{m+1}{m'+1}}L_{m+1}^{(m'-m)}\left[\epsilon_a\epsilon_b\frac{\epsilon_b\alpha_a + \epsilon_a\alpha_b}{\epsilon_a\alpha_a + \epsilon_b\alpha_b}|\xi|^2\right]G_m(t)G_{m'}^*(t).$$

Consequently, the function $\tilde{\mathrm{K}}_\xi(\alpha_a,\alpha_b;t)$ which appears in the integrand of $\chi(\xi;t)$ can be determined as follows:

$$\tilde{\mathrm{K}}_\xi(\alpha_a,\alpha_b;t) = \sum_{m,m'=0}^{\infty}\Upsilon_\xi^{(m,m')}(\alpha_a,\alpha_b)\Gamma_\xi^{(m,m')}(\alpha_a,\alpha_b;t) \tag{B.5}$$

with $\Gamma_\xi^{(m,m')}(\alpha_a,\alpha_b;t) = \mathrm{I}_\xi^{(m,m')}(\alpha_a,\alpha_b;t) + \mathrm{J}_\xi^{(m,m')}(\alpha_a,\alpha_b;t)$.

An immediate application of this result is the calculation of $\mathrm{K}_\gamma(\alpha_a,\alpha_b;t)$ since both the functions are connected by a two-dimensional Fourier transform. Now, substituting (B.5) into equation (22) and integrating in the complex variable ξ, we obtain as a result the analytical expression

$$\mathrm{K}_\gamma(\alpha_a,\alpha_b;t) = \sum_{m,m'=0}^{\infty}\mathrm{C}_\gamma^{(m,m')}(\alpha_a,\alpha_b)\mathrm{M}_\gamma^{(m,m')}(\alpha_a,\alpha_b;t) \tag{B.6}$$

where

$$\mathrm{C}_\gamma^{(m,m')}(\alpha_a,\alpha_b) = 2\exp(-|\epsilon_a\alpha_a + \epsilon_b\alpha_b|^2 - 2\gamma_a\gamma_b^*)[2\epsilon_a(\epsilon_a\alpha_a + \epsilon_b\alpha_b)^*\gamma_a]^{m'-m} \times \frac{\left[\left(\epsilon_a^2 - \epsilon_b^2\right)(\epsilon_a\alpha_a - \epsilon_b\alpha_b)(\epsilon_a\alpha_a + \epsilon_b\alpha_b)^*\right]^m}{m'!}$$

and

$$\mathrm{M}_\gamma^{(m,m')}(\alpha_a,\alpha_b;t) = L_m^{(m'-m)}\left[-\frac{4\epsilon_a\epsilon_b}{\epsilon_a^2 - \epsilon_b^2}\frac{\epsilon_b\alpha_a + \epsilon_a\alpha_b}{\epsilon_a\alpha_a - \epsilon_b\alpha_b}\gamma_a\gamma_b^*\right]F_m(t)F_{m'}^*(t) + \sqrt{\frac{m+1}{m'+1}} \times \left(\epsilon_a^2 - \epsilon_b^2\right)\frac{\epsilon_a\alpha_a - \epsilon_b\alpha_b}{\epsilon_a\alpha_a + \epsilon_b\alpha_b}L_{m+1}^{(m'-m)}\left[-\frac{4\epsilon_a\epsilon_b}{\epsilon_a^2 - \epsilon_b^2}\frac{\epsilon_b\alpha_a + \epsilon_a\alpha_b}{\epsilon_a\alpha_a - \epsilon_b\alpha_b}\gamma_a\gamma_b^*\right]G_m(t)G_{m'}^*(t)$$

with $\gamma_{a(b)} = \gamma - \epsilon_{a(b)}(\epsilon_{a(b)}\alpha_a - \epsilon_{b(a)}\alpha_b)$. In particular, when $\kappa_{a(b)} = \kappa$ the expression for $\mathrm{K}_\gamma(\alpha_a,\alpha_b;t)$ can be written in the simplified form

$$\mathrm{K}_\gamma(\alpha_a,\alpha_b;t) = \sum_{m,m'=0}^{\infty}\mathrm{O}_\gamma^{(m)}(\alpha_a,\alpha_b)\left[\mathrm{O}_\gamma^{(m')}(\alpha_a,\alpha_b)\right]^*\mathcal{R}_\gamma^{(m,m')}(\alpha_a,\alpha_b;t) \tag{B.7}$$

where

$$\mathrm{O}_\gamma^{(m)}(\alpha_a,\alpha_b) = \sqrt{2}\exp\left[-\frac{1}{4}(|\alpha_a + \alpha_b|^2 + |2\gamma - (\alpha_a - \alpha_b)|^2)\right]\frac{\{(\alpha_a + \alpha_b)[2\gamma - (\alpha_a - \alpha_b)]^*\}^m}{2^m m!}$$

$$\mathcal{R}_\gamma^{(m,m')}(\alpha_a,\alpha_b;t) = F_m(t)F_{m'}^*(t) + \frac{1}{2}|2\gamma - (\alpha_a - \alpha_b)|^2\frac{G_m(t)G_{m'}^*(t)}{\sqrt{(m+1)(m'+1)}}.$$

This solution is equivalent to considering that the interaction between atom and cavity (external) field has the same strength. Note that (B.6) represents an important step in the

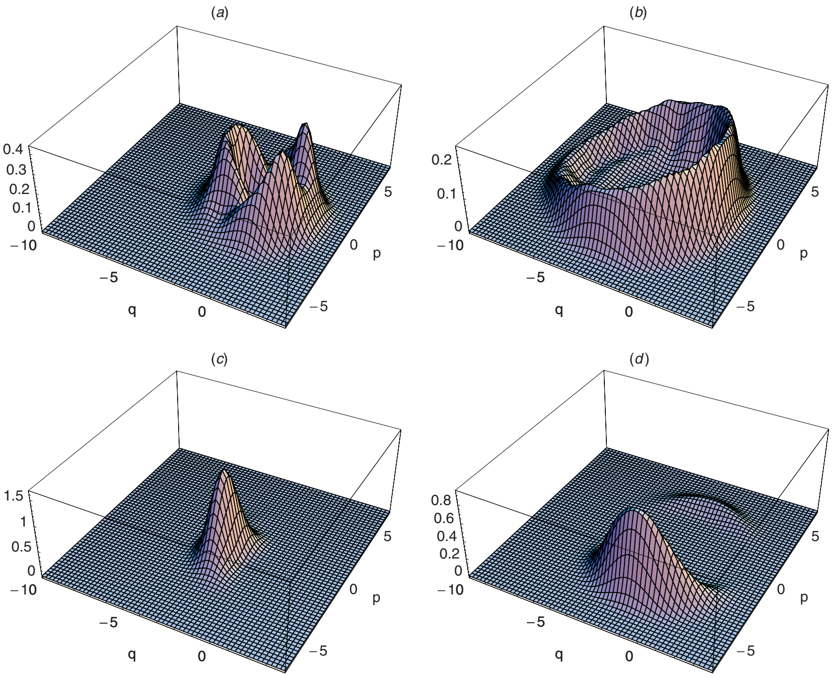

Figure 6. Plots of $W_a^{(c)}(\gamma; t) = \mathrm{K}_\gamma(\alpha, \beta; t)$ versus $p \in [-7, 7]$ and $q \in [-10, 4]$ for the atom–cavity system resonant (*a*), (*b*) $\delta = 0$ (maximum entanglement) and nonresonant (*c*), (*d*) $\delta = 10\kappa_{\mathrm{eff}}$ (minimum entanglement), with $|\alpha| = 1$ and $\kappa_{\mathrm{eff}} t = 100$ fixed. We also have considered two different values of amplitude of the external quantum field: (*a*), (*c*) $|\beta| = 2$ and (*b*), (*d*) $|\beta| = 5$, where the condition $\kappa_{\mathrm{a(b)}} = \kappa$ was established in both situations.

process of investigation of the effects due to the amplitude of the external quantum field and the detuning parameter on the nonclassical properties of the cavity field via the Wigner function.

For instance, when the external and cavity fields were described by coherent states, the Wigner function coincides with $\mathrm{K}_\gamma(\alpha, \beta; t)$ and for $t = 0$, we obtain the initial Wigner function $W_a^{(c)}(\gamma; 0) = 2\exp(-2|\gamma - \alpha|^2)$. Figure 6 shows the three-dimensional plots of $W_a^{(c)}(\gamma; t)$ versus $p = \sqrt{2}\,\mathrm{Im}(\gamma)$ and $q = \sqrt{2}\,\mathrm{Re}(\gamma)$ considering the atom–cavity system resonant (*a*), (*b*) $\delta = 0$ and nonresonant (*c*), (*d*) $\delta = 10\kappa_{\mathrm{eff}}$, for $|\alpha| = 1$ ($\langle \mathbf{n}_\mathrm{a} \rangle_\mathrm{c} = 1$) and $\kappa_{\mathrm{eff}} t = 100$ fixed, with two different values of amplitude of the external field: (*a*), (*c*) $|\beta| = 2$ ($\langle \mathbf{n}_\mathrm{b} \rangle_\mathrm{c} = 4$) and (*b*), (*d*) $|\beta| = 5$ ($\langle \mathbf{n}_\mathrm{b} \rangle_\mathrm{c} = 25$). The condition $\kappa_{\mathrm{a(b)}} = \kappa$ was established in both situations, and the infinite sums present in (B.7) were substituted by finite sums as follows:

$$\mathrm{K}_\gamma(\alpha, \beta; t) = \sum_{m=0}^{\ell} \left| \mathrm{O}_\gamma^{(m)}(\alpha, \beta) \right|^2 \mathcal{R}_\gamma^{(m,m)}(\alpha, \beta; t) + 2\,\mathrm{Re}\left[\sum_{m=0}^{\ell-1} \sum_{m'=m+1}^{\ell} \mathrm{O}_\gamma^{(m)}(\alpha, \beta) \left[\mathrm{O}_\gamma^{(m')}(\alpha, \beta) \right]^* \mathcal{R}_\gamma^{(m,m')}(\alpha, \beta; t) \right]$$

where ℓ is the maximum value which guarantees the convergence of this expression (we have fixed $\ell = 50$ in the numerical investigations). Since the time evolution of composite systems leads us to the essential concept of entanglement [5], figures 6(*a*)–(*d*) reflect the effects of the external quantum field on the different forms of entanglement in the tripartite system for a specific value of $\kappa_{\text{eff}} t$ (maximum entanglement when $\delta = 0$, and minimum entanglement for $\delta = 10\kappa_{\text{eff}}$). For a global view of entanglement of the system under consideration, different values of $\kappa_{\text{eff}} t$ and $(|\beta|, \delta)$ are necessary. Here, we give only a partial view of this important effect.

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